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Approximation of x^n by Reciprocals of Polynomials

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We consider the problem of approximating x^n by reciprocals of polynomials on the interval [0, 1]. We derive precise estimates for

$$\inf_{p_m \in \Pi_m} \|x^n - 1/p_m(x)\|_{[0,1]}$$

where Π_m denotes the set of all algebraic polynomials of degree at most m and where $\|\cdot\|_{[a,b]}$ denotes the supremum norm on $[a, b]$. For the case $m = n$ we show that

$$\left(\inf_{p_n \in \Pi_n} \|x^n - 1/p_n(x)\|\right)^{1/n} \to 27/64. \tag{1}$$

This sharpens estimates derived by Newman in [2] and answers question 7 posed by Reddy in [4]. Our method is to first solve an easier approximation problem using known L^2 results. This is similar to the approach used by Schönhage [5] to approximate e^{-x} on $[0, \infty)$ and by Rahman and Schmeisser [3] to approximate x^{-n} on $[1, \infty)$.

Our main result is the following:

THEOREM 1. *There exists $p_m \in \Pi_m$ so that*

$$\|x^n - 1/p_m(x)\|_{[0,1]} \leqslant \frac{(4.72)\, n}{(2n-1)^{1/2}} \cdot \frac{(n+m)!\,(3n-2)!}{(n-1)!\,(3n+m-1)!}.$$

For each $q_m \in \Pi_m$

$$\|x^n - 1/q_m(x)\|_{[0,1]} \geqslant \frac{(0.18)}{(2n+1)^{1/2}} \cdot \frac{(n+m)!\,(3n)!}{(n-1)!\,(3n+m+1)!}.$$

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The related least-squares result is

THEOREM 2. [1, p. 196] *Let* $n, \alpha_0, \ldots, \alpha_m$ *be distinct positive real numbers. The least-squares distance on* $[0, 1]$ *from* x^n *to the subspace spanned by* $\{x^{\alpha_0}, \ldots, x^{\alpha_m}\}$ *is*

$$\frac{1}{\sqrt{2n+1}} \prod_{i=0}^{m} \frac{|n-\alpha_i|}{n+\alpha_i+1}.$$

This result allows us to deduce the supremum norm bounds in the next theorem.

THEOREM 3. *There exists* $p_m \in \Pi_m$ *so that*

$$\|x^{2n} \cdot p_m(x) - x^n\|_{[0,1]} \leqslant \frac{n}{(2n-1)^{1/2}} \cdot \frac{(n+m)!\,(3n-2)!}{(n-1)!\,(3n+m-1)!}.$$

For each $q_m \in \Pi_m$,

$$\|x^{2n} \cdot q_m(x) - x^n\|_{[0,1]} \geqslant \frac{1}{(2n+1)^{1/2}} \cdot \frac{(n+m)!\,(3n)!}{(n-1)!\,(3n+m+1)!}.$$

Proof. The proof is identical to that used in deducing the supremum norm version of Müntz's theorem from the L^2 version (see [1, p. 197]). The lower bound is immediate from Theorem 2 with

$$\alpha_i = 2n + i \qquad \text{for } i = 0, 1, \ldots, m.$$

To derive the upper bound we observe, as in [1, p. 198], that

$$\left\| x^n - \sum_{j=2n}^{2n+m} \lambda_j x^j \right\|_{[0,1]} \leqslant \left(\int_0^1 \left| n t^{n-1} - \sum_{j=2n}^{2n+m} j \lambda_j t^{j-1} \right|^2 dt \right)^{1/2}.$$

We now apply Theorem 2 with

$$\alpha_i = 2n - 1 + i \qquad \text{for} \quad i = 0, 1 \ldots, m.$$

The remainder of the paper is concerned with deriving Theorem 1 from Theorem 3. We need the following somewhat technical lemmas.

LEMMA 1. *Let* $a > 0$ *and let* $1/p_m$ *be the best uniform approximation to* x^n *on* $[a, b]$ *from the set of reciprocals of elements of* Π_m. *Then* p_m *is decreasing on* $[0, a]$.

Proof. We know [1, p. 161] that $1/p_m$ interpolates x^n at $m+1$ points on $[a, b]$. By Descartes' rule of signs, since $1 - x^n \cdot p_m$ has $m+1$ zeroes on $[a, b]$, we deduce that $p_m(x) = \alpha x^m +$ lower order terms, where

$$\alpha = (-1)^m |\alpha| \neq 0.$$

Thus, there exist $a \leqslant \alpha_1 \leqslant \cdots \leqslant \alpha_m \leqslant b$ so that

$$(-1 + x^n \cdot p_m(x))' = x^{n-1}(n+m)\,\alpha \prod_{i=1}^{m} (x - \alpha_i)$$

and

$$(x \cdot p_m'(x) + np_m(x)) = (n+m)(-1)^m |\alpha| \prod_{i=1}^{m} (x - \alpha_i). \tag{2}$$

Suppose that there exist points c_1 and c_2 with $0 < c_1 < c_2 < \alpha_1$ so that $p_m'(c_1) = p_m'(c_2) = 0$. Then, from (2) it follows that

$$p_m(c_2) < p_m(c_1) < p_m(0). \tag{3}$$

We note that $p_m(\alpha_1) > 0$ and hence, by (2), $p_m'(\alpha_1) < 0$. Thus, the maximum of p_m on the interval $[0, \alpha_1]$ occurs at 0. It also follows from the above that the only way p_m can have a local max. or min. in $(0, \alpha_1)$ is if, in fact, p_m' has two zeros $0 < d_1 < d_2 < \alpha_1$ so that $p_m(d_1) < p_m(d_2)$. This, however, contradicts (3) and we see that p_m must be decreasing on $[0, \alpha_1]$.

LEMMA 2. *If there exists $p_m \in \Pi_m$ so that*

$$\|(p_m(x) - x^{-n})\, x^{2n}\|_{[\rho + \rho/n,\, 1]} \leqslant \rho^n$$

then there exists $q_m \in \Pi_m$ so that

$$\|x^n - 1/q_m(x)\|_{[0,1]} \leqslant (2+e)\,\rho^n.$$

Proof. For $x \in [\rho + \rho/n, 1]$

$$|p_m(x) - x^{-n}| \leqslant \frac{\rho^n}{(\rho + \rho/n)^n x^n} \leqslant \frac{1}{2x^n}$$

and

$$x^n \cdot p_m(x) \geqslant \tfrac{1}{2}.$$

Thus,

$$\begin{aligned}
&\|x^n - 1/p_m(x)\|_{[\rho+\rho/n,\,1]}\\
&\quad = \left\| (p_m(x) - x^{-n}) \left(\frac{x^{2n}}{x^n \cdot p_m(x)} \right) \right\|_{[\rho+\rho/n,\,1]}\\
&\quad \leqslant 2\rho^n. \qquad (4)
\end{aligned}$$

Suppose that $1/q_m$ is the best approximation to x^n on $[\rho + \rho/n, 1]$ from the reciprocals of elements of Π_m. From the previous lemma we see that q_m is decreasing on $[0, \rho + \rho/n]$ and hence,

$$\|x^n - 1/q_m(x)\|_{[0,\,\rho+\rho/n]} \leqslant \max. \left(\frac{1}{q_m(\rho+\rho/n)}, (\rho+\rho/n)^n \right)$$

From (4) and the above we have

$$\begin{aligned}
\frac{1}{q_m(\rho+\rho/n)} &\leqslant \|x^n - 1/q_m(x)\|_{[\rho+\rho/n,\,1]} + (\rho+\rho/n)^n\\
&\leqslant 2\rho^n + (\rho+\rho/n)^n \leqslant (2+e)\,\rho^n
\end{aligned}$$

and

$$\|x^n - 1/q_m(x)\|_{[0,\,1]} \leqslant (2+e)\,\rho^n.$$

LEMMA 3. *If there exists $p_m \in \Pi_m$ so that*

$$\|x^{2n} \cdot p_m(x) - x^n\|_{[\rho+\rho/n,\,1]} \leqslant \rho^n$$

then there exists $q_m \in \Pi_m$ so that

$$\|x^{2n} \cdot q_m(x) - x^n\|_{[0,\,1]} \leqslant e\,\rho^n.$$

Proof. Let $q_m \in \Pi_m$ satisfy

$$\|x^{2n} \cdot q_m(x) - x^n\|_{[\rho+\rho/n,\,1]} = \min_{p_m \in \Pi_m} \|x^{2n} \cdot p_m - x^n\|_{[\rho+\rho/n,\,1]}\,.$$

As in the proof of Lemma 1, if $q_m(x) = \alpha x^m + \cdots$, then

$$(x^n \cdot q_m(x))' = x^{n-1}(n+m)(-1)^m |\alpha| \prod_{i=1}^{m} (x - \alpha_i), \qquad \alpha_i \in [\rho + \rho/n, 1].$$

It follows that $x^n \cdot q_m(x)$ is non-decreasing and positive on $[0, \rho + \rho/n]$ and that

$$\|x^n \cdot q_m(x) - 1\|_{[0, \rho+\rho/n]} \leqslant \max\left(1, \frac{\rho^n}{(\rho + \rho/n)^n}\right) \leqslant 1.$$

We complete the result by noting that this shows that

$$\|x^{2n} \cdot q_n(x) - x^n\|_{[0, \rho+\rho/n]} \leqslant (\rho + \rho/n)^n \leqslant e\,\rho^n.$$

Proof of Theorem 1. Theorem 3 guarantees the existence of $p_m \in \Pi_m$ so that

$$\|x^{2n} \cdot p_m(x^n) - x^n\|_{[0,1]} \leqslant \frac{n}{(2n-1)^{1/2}} \cdot \frac{(n+m)!\,(3n-2)!}{(n-1)!\,(3n+m-1)!} = \delta^n.$$

Thus,

$$\|x^{2n} \cdot p_m(x) - x^n\|_{[\delta+\delta/n, 1]} \leqslant \delta^n$$

and by Lemma 2, there exists $q_m \in \Pi_m$ so that

$$\|x^n - 1/q_m(x)\|_{[0,1]} \leqslant (2+e)\,\delta^n.$$

We now establish the lower bound. We know by Theorem 3, for all $p_m \in \Pi_m$

$$\|x^{2n} \cdot p_m(x) - x^n\|_{[0,1]} \geqslant \frac{1}{(2n+1)^{1/2}} \cdot \frac{(n+m)!\,(3n)!}{(n-1)!\,(3n+m+1)!} = \rho^n.$$

By Lemma 3,

$$\|x^{2n} \cdot p_m(x) - x^n\|_{[\rho+\rho/n, 1]} \geqslant \frac{\rho^n}{e}. \tag{5}$$

This, as we shall show, finishes the proof by implying that

$$\|x^n - 1/p_m(x)\|_{[\rho+\rho/n, 1]} \geqslant \frac{\rho^n}{2e}.$$

This final inequality can be seen as follows.
Suppose, for $n > 1$, that

$$\|x^n - 1/p_m(x)\|_{[\rho+\rho/n, 1]} < \frac{\rho^n}{2e}.$$

Then, for $x \in [\rho + \rho/n, 1]$,

$$\frac{1}{p_m(x)} > x^n - \frac{\rho^n}{2e} \geqslant \frac{x^n}{2}$$

and

$$x^n \cdot p_m(x) \leqslant 2.$$

This, however, contradicts (5) by implying that

$$\begin{aligned} &\| x^{2n} \cdot p_m(x) - x^n \|_{[\rho + \rho/n, 1]} \\ &\quad = \| (x^n - 1/p_m(x))(x^n \cdot p_m(x) \|_{[\rho + \rho/n, 1]} \\ &\quad < \frac{\rho^n}{e}. \end{aligned}$$

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